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A Uniform Constant of Strong Uniqueness on an Interval

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Consider approximation in $C[\alpha, \beta]$ with respect to the Chebyshev (sup) norm. Real continuous functions f are to be approximated by elements of a fixed n-dimensional Haar subspace H. It is well known [2] that there is $\gamma = \gamma(f) > 0$, called a constant of strong uniqueness (SU constant), such that for p^* best to f,

$$\|f - p\| - \|f - p^*\| \geqslant \gamma \|p - p^*\|,$$

for all $p \in H$. In this note we consider whether certain subsets of $C[\alpha, \beta]$ have a *uniform* $\gamma > 0$.

Such a problem was considered by Cline [3] for a general compact X (rather than for an interval $[\alpha, \beta]$), by Bartelt [1] for finite domains, and by Henry and Schmidt [5] for intervals. The subsets of Henry and Schmidt are compact and hence include no neighborhoods: our subsets will not have that defect.

A uniform constant of strong uniqueness guarantees a uniform Lipschitz constant, as shown in the text of Cheney [2, p. 82].

In approximation by constants it is seen that we can set $\gamma = 1$. More generally, in approximation by Haar subspace of dimension 1, it is seen by arguments as in [2, p. 81] that γ can be set equal to

$$\inf\{\phi(x): x \in [\alpha, \beta]\}/\sup\{\phi(x): x \in [\alpha, \beta]\}$$

for ϕ, a positive basis function. It has been proven by Cline [3, p. 164] that a uniform Lipschitz constant cannot exist on infinite compact X when $n > 1$: by the cited result in Cheney no uniform SU constant can exist in this case, either. As Cline's construction is elaborate, it may be more instructive to consider a simple example.

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EXAMPLE. Let $[\alpha, \beta] = [-1, 1]$. Approximate by first-degree polynomials. Let

$$f_k(0) = 1 \qquad f_k(1/k) = -1 \qquad f_k(x) = 0, \text{ for } x \geq 2/k.$$

Extend f_k to $[0, 1]$ by straight lines between 0 and $1/k$ and between $1/k$ and $2/k$. Extend f_k to $[-1, 1]$ by making f_k even. As $f_k - 0$ alternates twice on $[-1, 1]$, 0 is the unique best approximation to f_k with error norm 1. The approximation ηx has an error norm of $1 - \eta/k$ for η small and differs from 0 by η in norm. Hence a strong uniqueness constant γ for f_k can be no larger than $1/k$.

It is easily seen that this example can be extended to approximation by polynomials of degree n by constructing even f_k with $n + 3$ alternations in a $1/k$ neighborhood of zero and zero outside a $2/k$-neighbourhood of zero.

A key property appears to be separation in alternants.

DEFINITION. The separation of a strictly increasing $n + 1$ point set $\{x_0, ..., x_n\}$ is

$$\min\{x_{i+1} - x_i : i = 0, ..., n - 1\}.$$

THEOREM 1. *Let $n \geq 2$. Let f_k be a sequence such that the optimal error curve for f_k has only one alternant X^k and the separation of X^k tends to zero. Then $\gamma(f_k) \to 0$.*

Proof. Let p_k be best to f_k. $\{X^k\}$ has an accumulation point X^0 of no more than n distinct points. By taking a subsequence, if necessary, we can assume $\{X^k\} \to X^0$. By taking a subsequence, if necessary, we can assume that two adjacent points of the alternants X^k coalesce to a point $z \in X^0$, say $\{x_0^k\} \to z$, $\{x_1^k\} \to z$. We now consider the number of distinct points in X^0. If it is $\leq n - 1$, select $p_0 \not\equiv 0$ vanishing on X_0. If it is exactly n, select $p_0 \not\equiv 0$ vanishing on all but one point $\neq z$. Let the subscript of that point be j. Let $\epsilon > 0$ be given. Let $W = \{x: p_0(x) = 0, x \in X^0\}$. There is a neighborhood N of W such that $|p_0(x)| < \epsilon \| p_0 \|$ for $x \in N$. Select k such that $\{x_0^k, x_1^k\} \subset N$. In the event $X^0 \sim W$ is nonempty, there is a neighborhood M of x_j^k such that $f_k - p_k$ is of constant sign on M. As the only extrema of $f_k - p_k$ are in X^k, there is a constant L such that for $|\eta| < L$, the norm of $f_k - p_k - \eta p_0$ on $[\alpha, \beta]$ must be attained on $N \cup M$. For $|\eta| < L$ and $x \in N$,

$$|f_k(x) - p_k(x) - \eta p_0(x)| < \|f_k - p_k\| + |\eta| \epsilon \|p_0\|.$$

In the evant $X^0 \sim W$ is nonempty, choosing η small and of the correct sign gives

$$|f_k(x) - p_k(x) - \eta p_0(x)| < \|f_k - p_k\| \qquad x \in M$$

Hence for η small and of the correct sign,

$$\| f_k - p_k - \eta p_0 \| \leqslant \| f_k - p_k \| - |\eta| \epsilon \| p_0 \|.$$

But $\| p_k - (p_k - \eta p_0) \| = |\eta| \| p_0 \|$, hence f_k has a strong uniqueness constant of no more than ϵ.

It follows that if we want a uniform strong uniqueness constant γ, we must take only a subset of $C[\alpha, \beta]$, with separation in alternants desirable.

DEFINITION. For $\delta > 0$, let F_δ be the subset of $C[\alpha, \beta]$ with at least one alternant of the optimal error curve having separation $\geqslant \delta$.

THEOREM 2. *F_δ has a uniform strong uniqueness constant γ.*

Proof. Let γ be the infimum of $e(q, X) = \max\{(-1)^i q(x_i) : i = 0,..., n\}$ over all approximants q of norm 1 and all strictly increasing $n + 1$ point sets $X = \{x_0, ..., x_n\}$ with separation $\geqslant \delta$. This infimum is taken over a compact set and e depends continuously on q, X. By arguments in the text of Cheney, γ must be > 0 and is a uniform strong uniqueness constant.

We consider stability of separation under perturbation.

THEOREM 3. *Let $\{f_k\} \to f$ and p_k be best to f_k. Let X^k be an alternant of $f_k - p_k$. Any accumulation point of X^k is an alternant of $f - p^*$.*

Proof. By continuity of the best approximation operator, $\{p_k\} \to p^*$, hence $\{f_k - p_k\} \to f - p^*$. Let x be in an accumulation point W for $\{X_k\}$. Then x must be an extremum of $f - p^*$. If $f - p^*$ did not alternate n times on W, then $f_k - p_k$ would not alternate n times on X^k for k large.

COROLLARY. *Let f have a unique alternant X of its optimum error $f - p^*$. Let $\{f_k\} \to f$ and p_k be best to f_k. Let X^k be an alternant of $f_k - p_k$, then $\{X^k\} \to X$.*

Conversely, if f does not have a unique alternant, we can select an alternant X and construct g arbitrarily close to f with p^* best and X the only extrema of $g - p^*$. One such construction in the case $f - p^* \not\equiv 0$ is to choose g_k such that

$$g_k(x) - p^*(x) = [f(x) - p^*(x)][1 - \mathrm{dist}(x, X)/k].$$

It follows that the (maximum) separation of alternants for nearby functions g can be much less than for f.

DEFINITION. $G_\delta = \{f\text{: all alternants have separation } > \delta\}$.

By the arguments of Theorems 2 and 3,

THEOREM 4. *G_δ has a uniform strong uniqueness constant and is an open subset of $C[\alpha, \beta]$.*

A straightforward extension of the problem is the case of mandatory endpoint zeros. Consider approximation by an n-dimensional Haar subspace with null set Z [7, p. 291] consisting of one or both endpoints (that is, the subspace is a Haar subspace on $[\alpha, \beta] \sim Z$ and vanishes identically on Z), with f also vanishing on Z. An alternating theory holds in this case also [7, p. 292]. The arguments of Cheney [2, pp. 89–81] for choosing a $\gamma > 0$ extend to this case with no additional arguments needed. We show later that there can be no uniform SU constant over f continuous on $[\alpha, \beta]$ and vanishing on Z.

In the definition of separation we add $x_0 - \alpha$ to the braces if $\alpha \in Z$ and $\beta - x_n$ to the braces if $\beta \in Z$. Theorems 1–4 need no changes. In the case $n = 1$ we choose alternants X^k such that both points tend to a point of Z and use the arguments of Theorem 1 to get SU constant $\to 0$.

A further generalization is to replace $[\alpha, \beta]$ by a compact metric space X, as in the text of Cheney, and to possibly replace Haar subspaces by Haar subspaces with null space Z [7, p. 291] with f required to vanish on Z. We replace alternants by $n + 1$ point critical point sets (Rice [11, p. 233] = $n + 1$ point IED sets (Dunham [8, p. 132]) = $n + 1$ point minimal H-sets (Geiger [9]) = $n + 1$ point primitive extremal signatures (Gutknecht [10]). The theorems extend easily, with the definition of separation for an $n + 1$ point set $\{x_i\}$ being

$$\min\{\rho(x_i, x_j), i \neq j, \rho(x_i, z), z \in Z\}.$$

A further generalization is to approximate two continuous functions f^+ and f^- simultaneously, $f^- \leq f^+$, as in Dunham [6]. In the case of no straddle point [6, p. 473, (3)], best approximation on $[\alpha, \beta]$ is characterized by a special kind of alternation. The theory of the text of Cheney [2, Chap. 3] can handle simultaneous approximation on a compact metric space X by creating two copies of X. Cheney's argument [2, pp. 80–81] gives strong uniqueness if no straddle points occurs. Theorems 1–3 extend, Theorem 4 holds with $\{f^-, f^+\}$ in the definition of G_δ required to have no straddle points. To show openness, suppose there existed $\{f_k^-\} \to f^-$, $\{f_k^+\} \to f_k^+$, $f_k^- \leq f_k^+$, with (f_k^-, f_k^+) having a straddle point at x_k. Assume that $\{x_k\} \to x_0$, then (f^-, f^+) has a straddle point at x_0.

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